3 - Matrices and Transformations

graphics is fun; graphics requires matrix math;

thus, matrix math must be fun

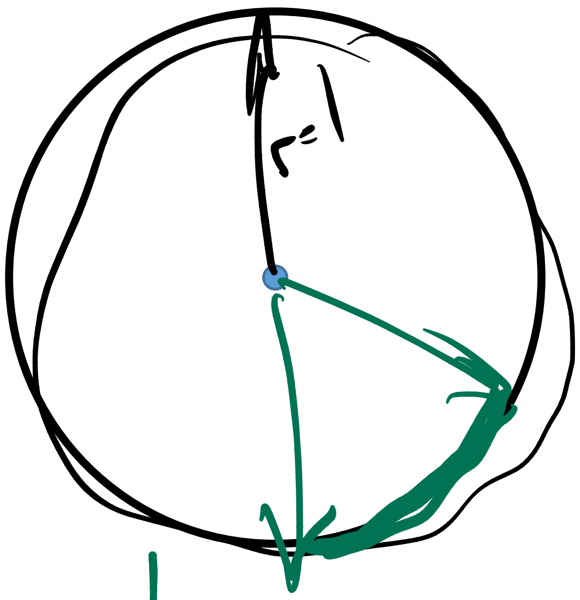
Readings

- Review **Math** (chapter 2) as needed
- **Matrices**: 5.2
- **Transformations**:
  - 6.0-6.1.5 (simple linear 2d transforms)
  - 6.2.0 (simple linear 3d transforms)
  - 6.3-6.5 (affine transformation, inverses of transformations, coordinate transformations)

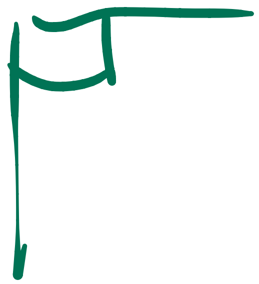
Simple Trig: Angles

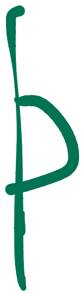
d
r
2π

circumference
radians
degrees



circ = 2π

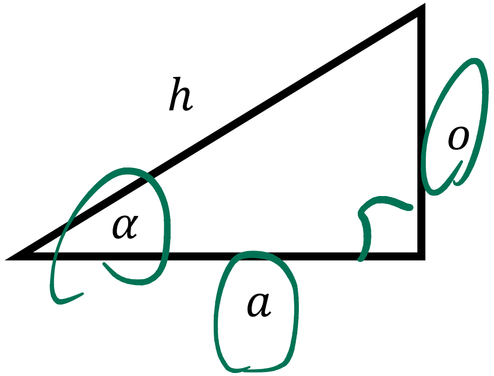
 $\pi/2$

 π

$$\text{deg} = \frac{\text{rad}}{\pi} \times 180$$

Simple Trig: Angles

$a^2 + o^2 = h^2$



$\sin = \frac{o}{h}$

$\tan \alpha = \frac{o}{a}$

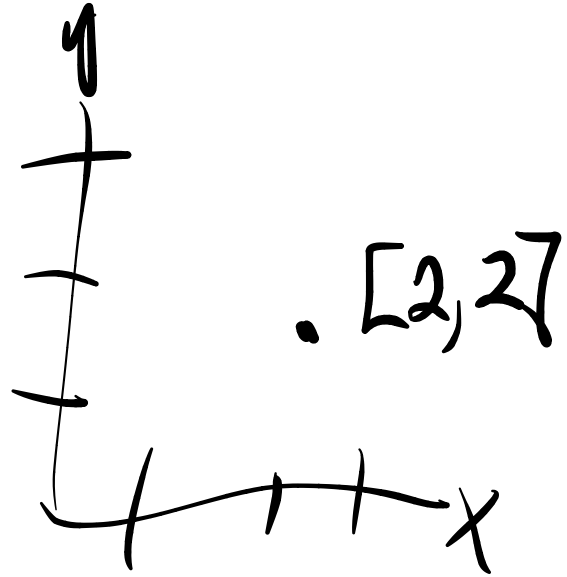
sin, cos, tan,
asin, acos, atan

2D Vectors

$$p = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$p_1 + p_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

$$p_1 s = \begin{bmatrix} x_1 s \\ y_1 s \end{bmatrix}$$

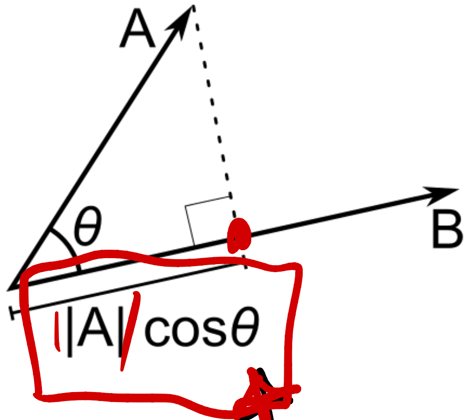


Dot Product

$$p_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \qquad p_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

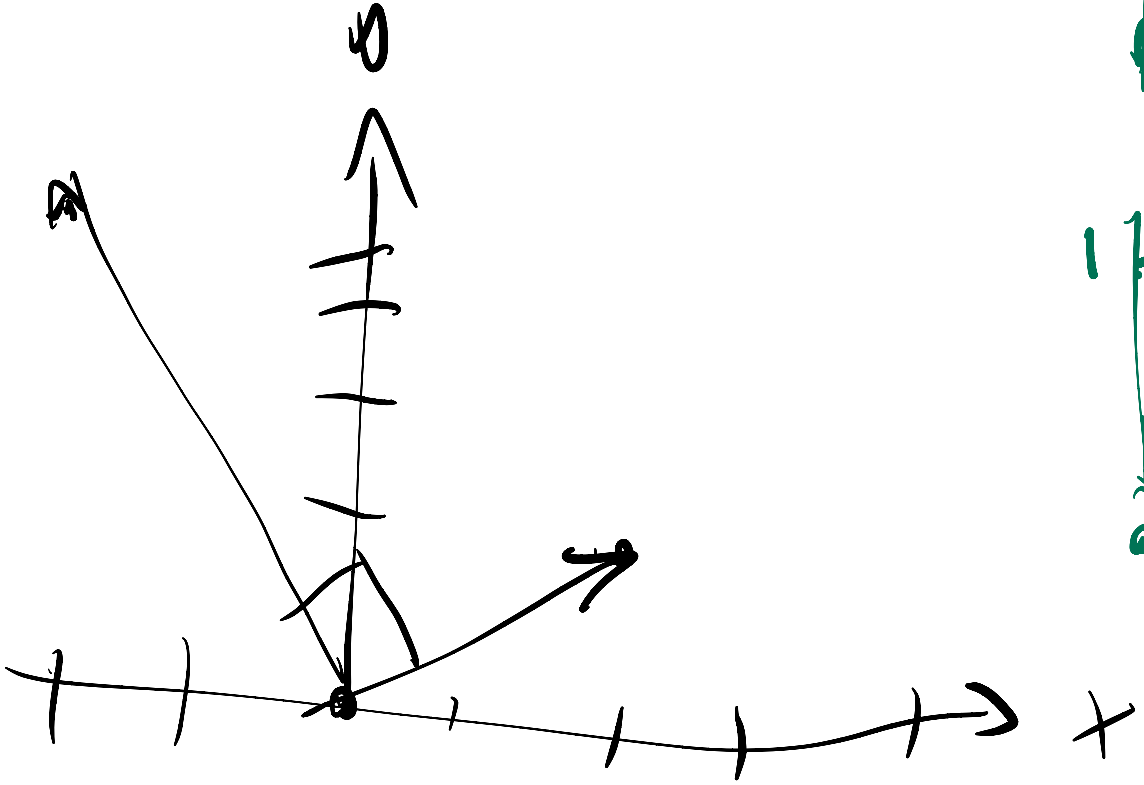
$$\mathbf{A} \cdot \mathbf{B} = \|\mathbf{A}\| \, \|\mathbf{B}\| \cos\theta,$$

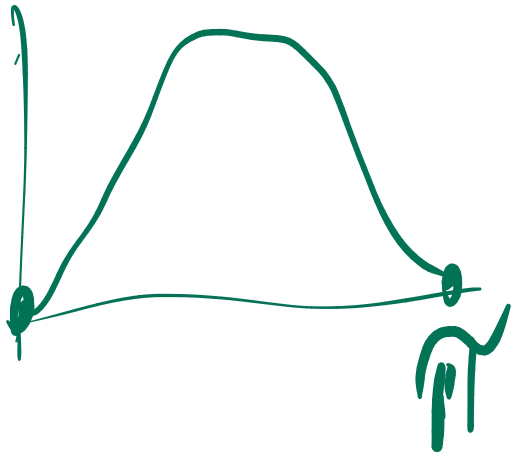
$$p_1 \cdot p_2 = \begin{bmatrix} x_1 \cdot x_2 + y_1 y_2 \end{bmatrix}$$

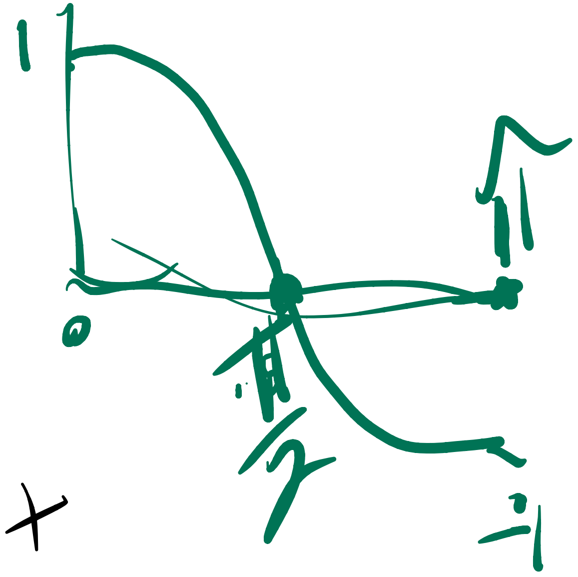


$$\|A\| = \text{Mag, length} = \sqrt{x^2 + y^2}$$

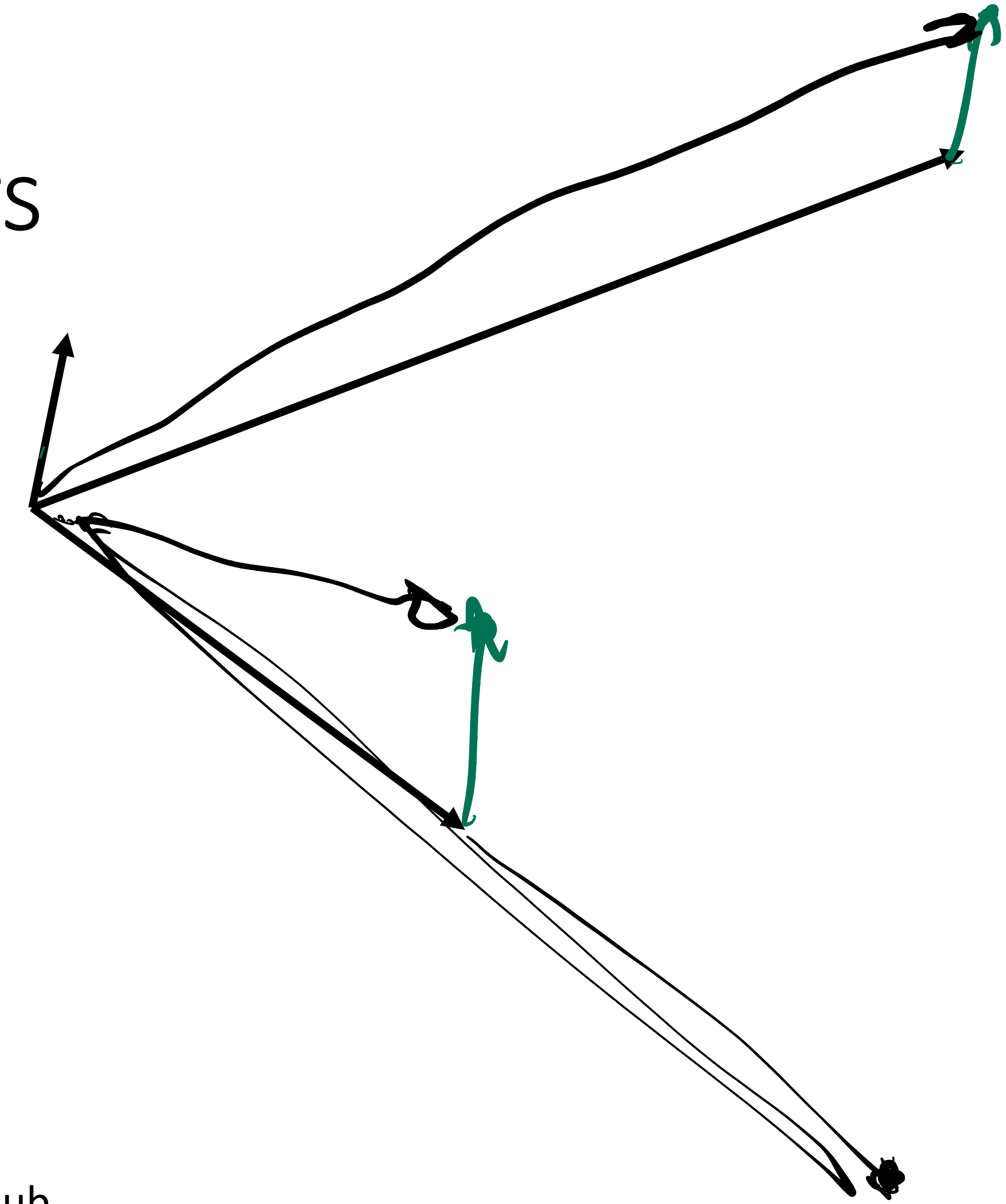
$$[2\ 1]\begin{bmatrix}-2\\4\end{bmatrix} = -4 + 4 = 0$$





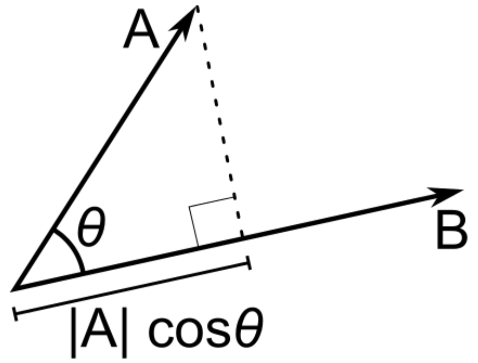


Vectors



add/sub
length
scalar mult

3-D Vectors



$$\mathbf{A} \cdot \mathbf{B} = \|\mathbf{A}\| \ \|\mathbf{B}\| \cos \theta,$$

Have length and direction

$$\mathbf{V = [x_v, y_v, z_v]}$$

Length is given by the Euclidean Norm

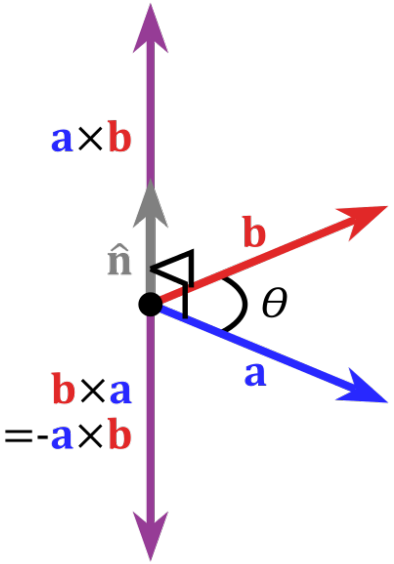
$$||V|| = \sqrt{(x_v^2 + y_v^2 + z_v^2)}$$

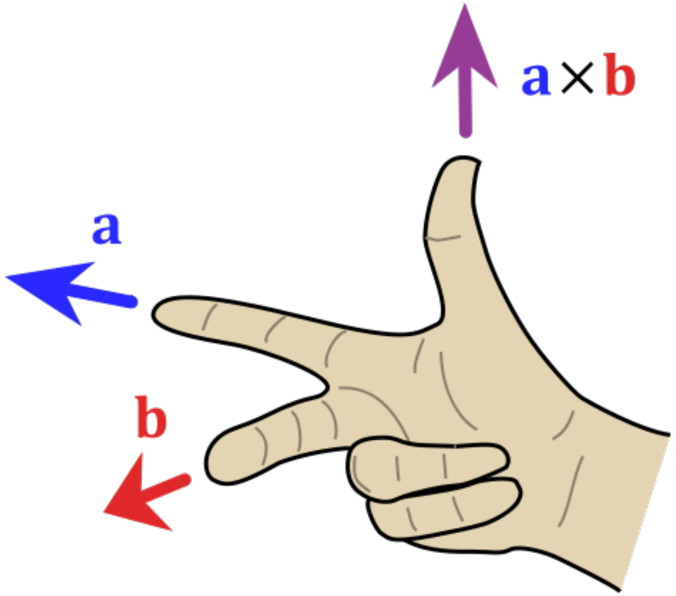
Dot Product $\mathbf{V \bullet U} = [x_v,y_v,z_v]\bullet[x_u,y_u,z_u]$

$$= x_v x_u + y_v y_u + z_v z_u$$

$$= ||V||\ ||U|| \cos ß$$

Cross Product



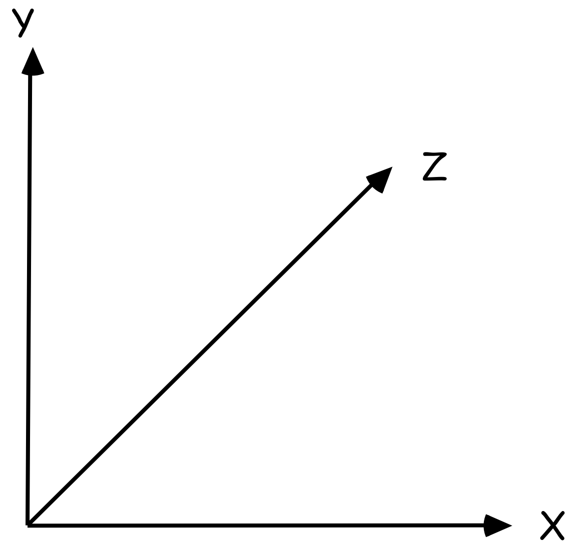


$$V \times U = [v_y u_z - v_z u_y,\ -v_x u_z + v_z u_x,\ v_x u_y - v_y u_x]$$

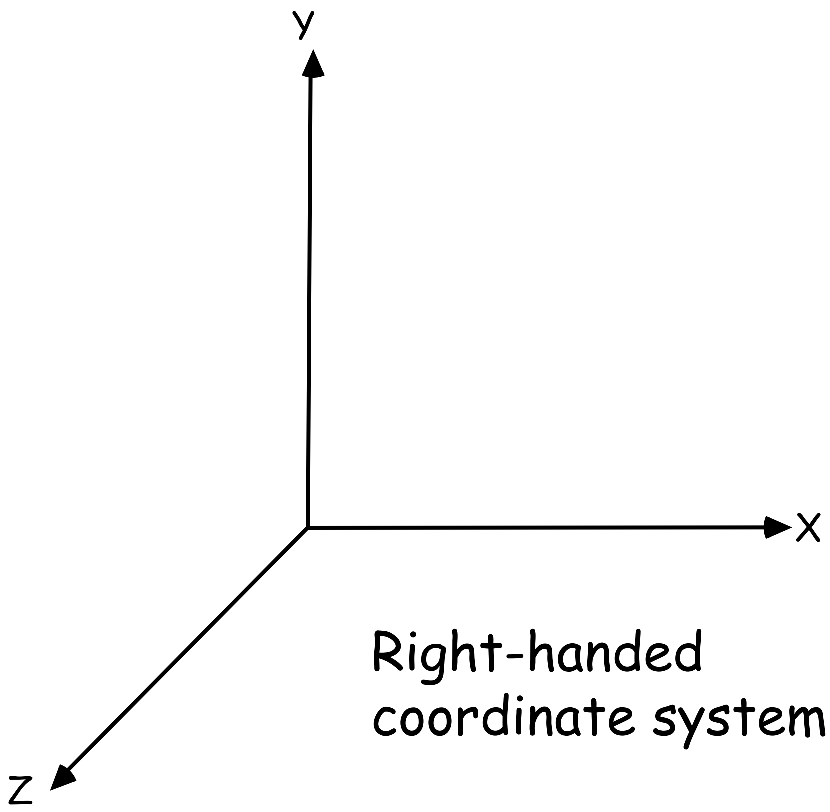
$$V \times U = -(U \times V)$$

Some images from https://en.wikipedia.org/wiki/Cross_product

3D Coordinate Systems



Left-handed
coordinate system



Right-handed
coordinate system

Matrices: Representation, Operations

Mult is not commutative
Identity
Inverses

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

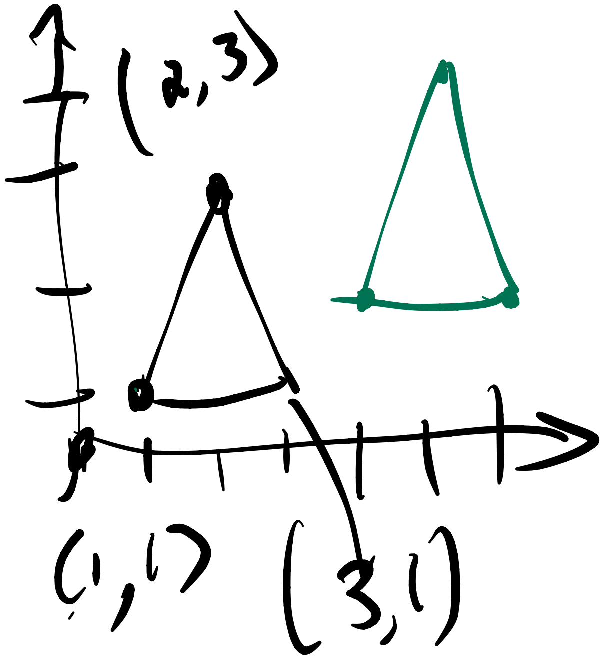
$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 23 \\ 24 \end{bmatrix}$$

Vector Operations with Matrices

Matrices as Transformations on Vectors

Translation: Change Position

move 3 in x
1 up in y

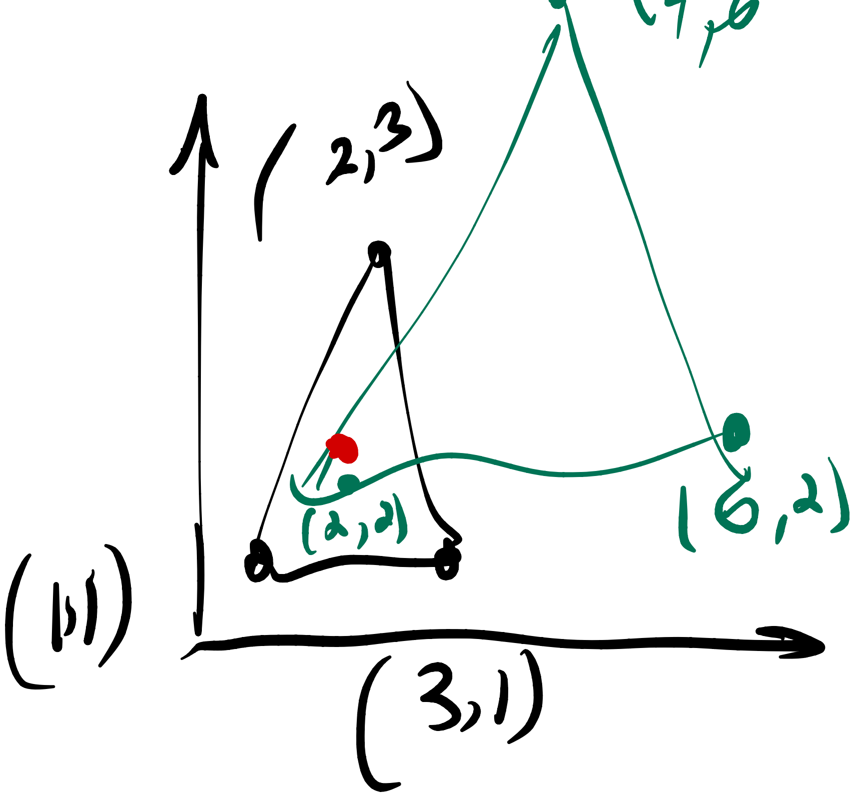


$x' = x + 3$
$y' = y + 1$

$P' = P + T$

$P = \begin{bmatrix} x \\ y \end{bmatrix}$ $P' = \begin{bmatrix} x' \\ y' \end{bmatrix}$ $T = \begin{bmatrix} d_x \\ d_y \end{bmatrix}$

Scale: Change Size



Scale by $s = 2$

$$x' = x \cdot s$$

$$y' = y \cdot s$$

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad P = \begin{bmatrix} x \\ y \end{bmatrix}$$

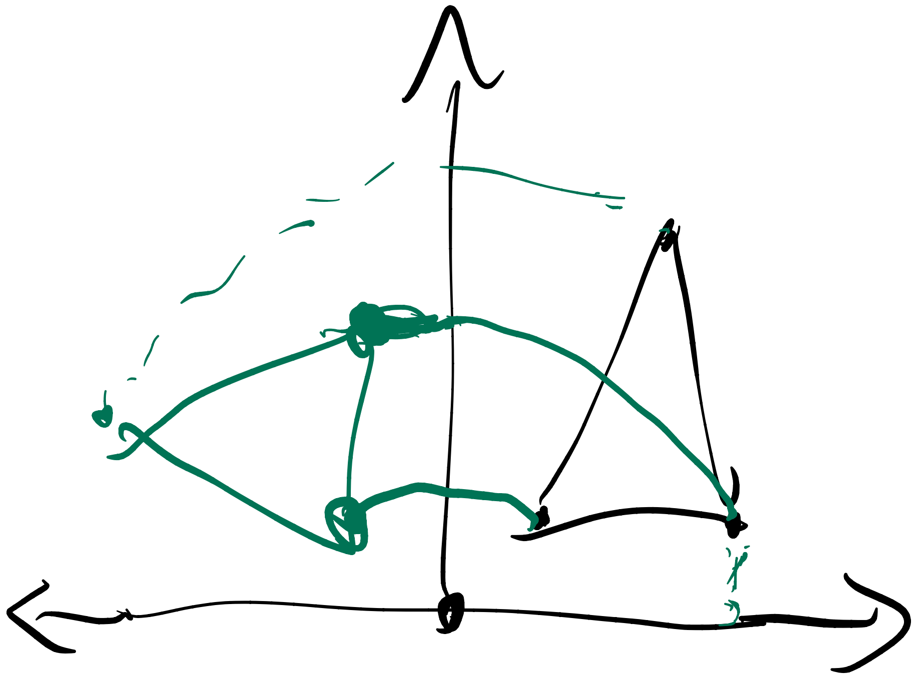
$$P' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$S = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \qquad S = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \qquad P' = \begin{bmatrix} sx + 0 \\ 0 + sy \end{bmatrix}$$

Rotation: Change Orientation

rotate 90°

counter-clockwise
[ccw]



$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$
$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = P'$$

Homogeneous Coordinates

points (x, y) $\quad P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ $\quad w \rightarrow 1$

$S = \begin{bmatrix} \boxed{\begin{matrix} S_x & \\ & S_y \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix} \quad S \cdot P = \begin{bmatrix} S_x x \\ S_y y \\ 1 \end{bmatrix}$

$P' = S \cdot P$

$P' = R \cdot P$

$P' = T \cdot P$

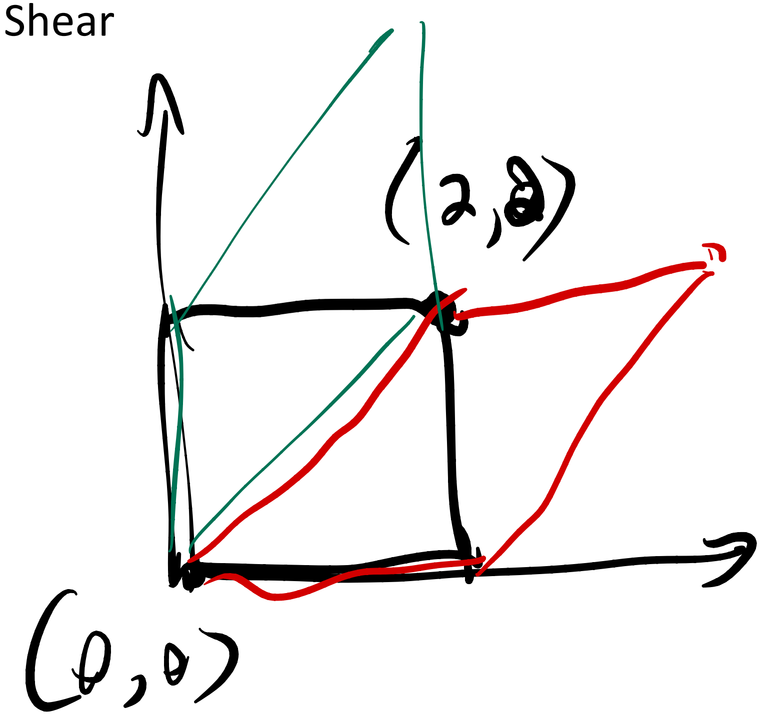
$c = \cos\theta$

$s = \sin\theta$

$R = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$T = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + d_x \\ y + d_y \\ 1 \end{bmatrix}$

Shear



$$x' = x + ay$$
$$y' = y$$

$a = 2$

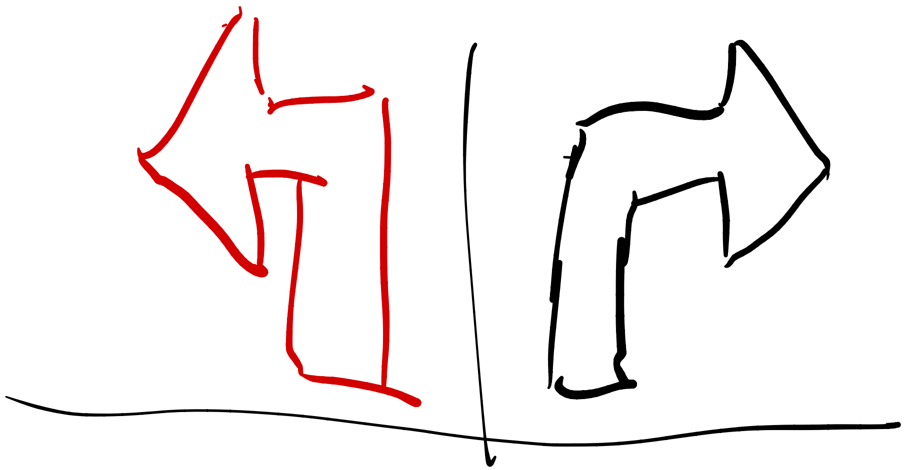
$$Sh_x = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{y}$

$$x' = x$$
$$y' = y + ax$$

$$Sh_y = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection



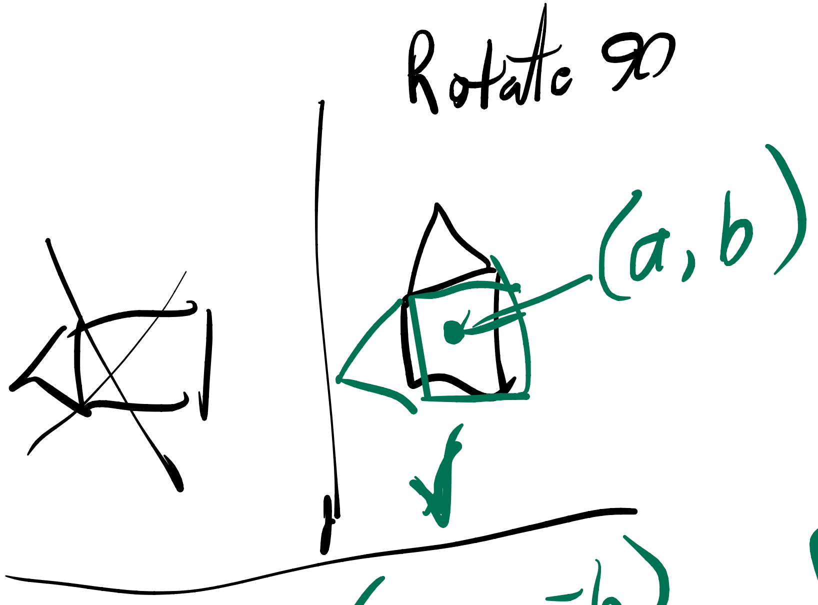
$$x' = -x$$
$$y' = y$$

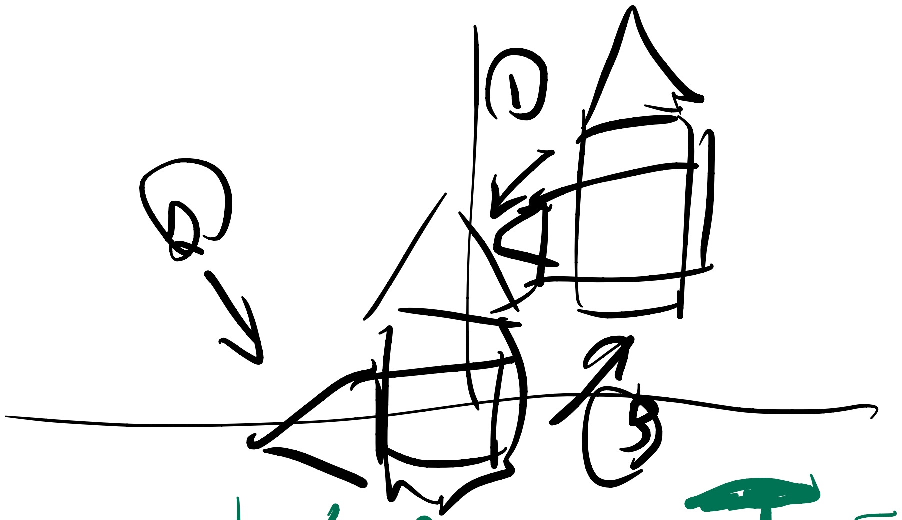
$S_x = -1$
$S_y = 1$

$$RF_x = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RF_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Composition of Transformations





$T_1 = tran(-a, -b)$

$$\begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$$

$R = rotate(90)$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T_2 = tran(a, b)$

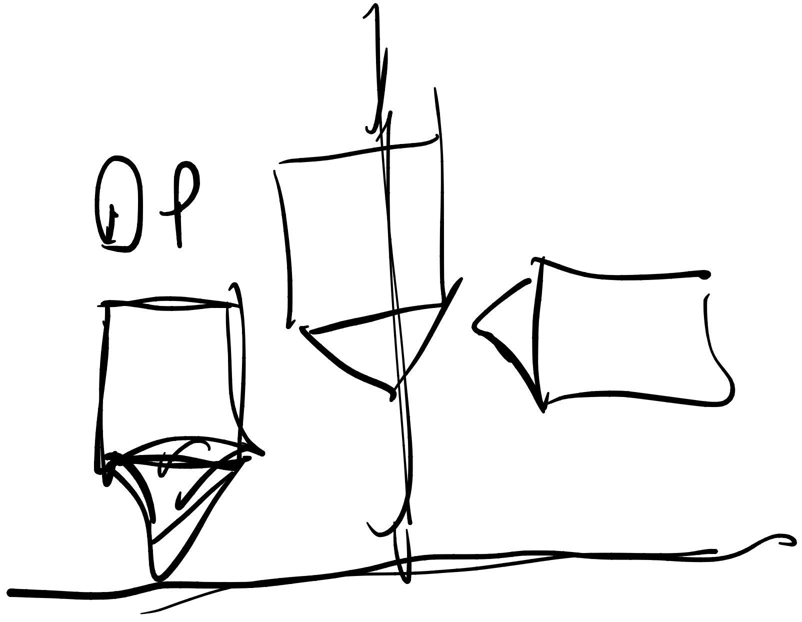
$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = T_2 \cdot R \cdot T_1 \cdot P$$

$$= T_2(R(T_1 \cdot P))$$

$$= (T_2 R T_1) P$$

$$? \neq (T_1 T_2 R) P$$

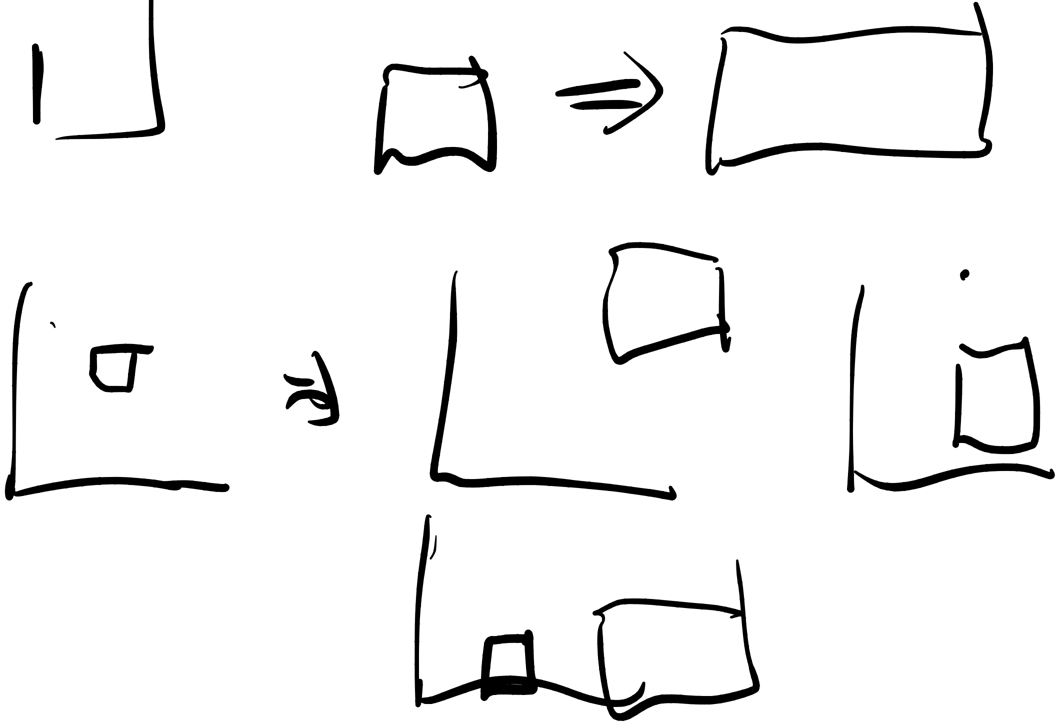


Commute?

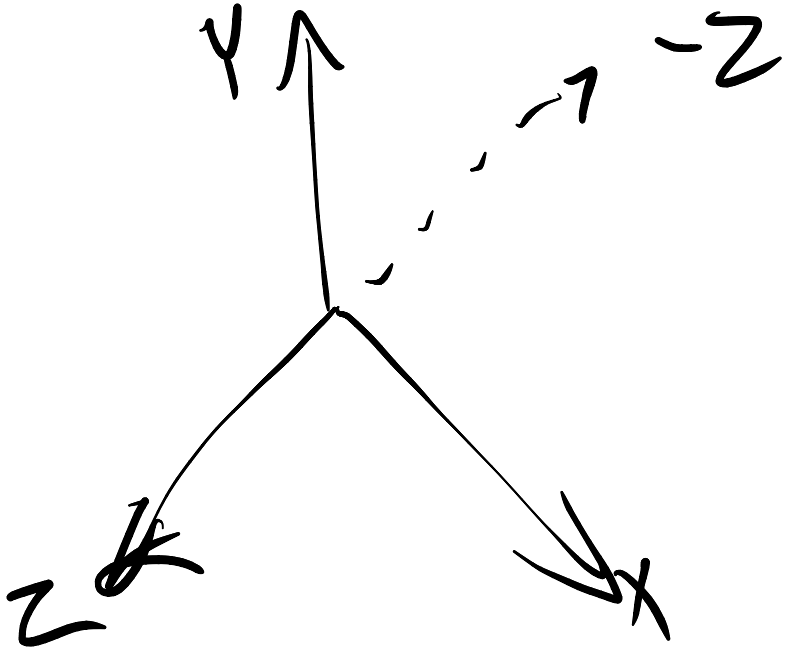
$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Non-uniform

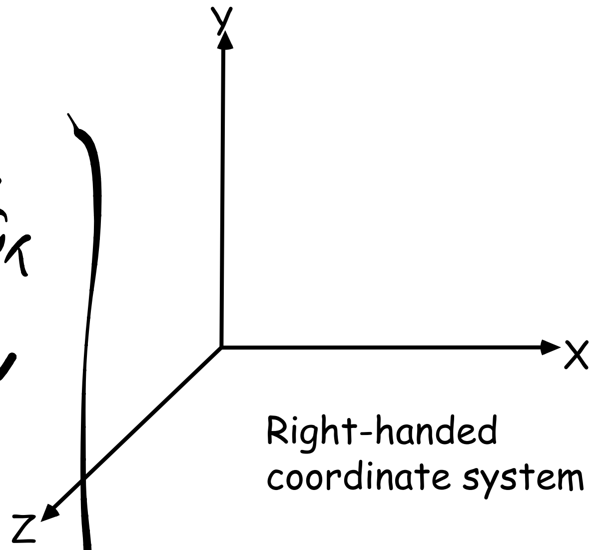
	T	R	US	NS
T	Y			
R	N	Y		
US	N	Y	Y	
NS	N	N	Y	Y



3D Transformations



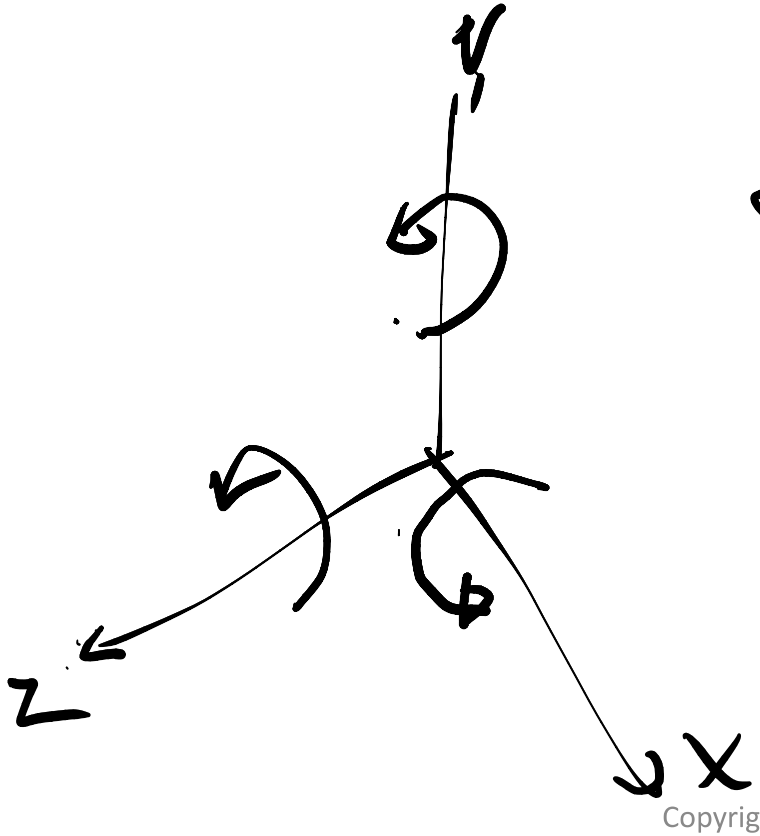
$$T = \begin{bmatrix} 1 & & & t_x \\ & 1 & & t_y \\ & & 1 & t_z \\ & & & 1 \end{bmatrix}$$



$$S = \begin{bmatrix} s_x & & & \\ & s_y & & \\ & & s_z & \\ & & & 1 \end{bmatrix}$$

Rotation

$$R_{x\theta} \begin{bmatrix} 1 & & & \\ & c & -s & \\ & s & c & \\ & & & 1 \end{bmatrix} \qquad R_z = \begin{bmatrix} c & -s & & \\ s & c & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$



$$R_y = \begin{bmatrix} c & & s & \\ & 1 & & \\ -s & & c & \\ & & & 1 \end{bmatrix}$$

R is NOT commutative in 3D

Graphics Libraries

https://eytanmanor.medium.com/the-story-of-webgpu-the-successor-to-webgl-bf5f74bc036a